**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH4230 2024-25**
**Lecture 14**
**February 26, 2025 (Wednesday)**

# 1 Recall

Yesterday, we discussed the property of subgradient and we left the proof for

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$$

as an after-class discussion. However, the proof for the " $\Longleftarrow$ " direction is difficult, so we may skip the general proof here and introduce the following theorem.

**Theorem 1.** *Let $f_1, f_2$ be convex functions, and*

$$\text{ri dom}(f_1) \cap \text{ri dom}(f_2) \neq \emptyset$$

*Then $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$, $\forall x \in \text{dom}(f_1) \cap \text{dom}(f_2)$.*

*Proof for simple case.* First, we claim that for convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. Then

$$\partial f(x) = [\partial_- f(x), \partial_+ f(x)]$$

where $\partial_- f(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}$ and $\partial_+ f(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}$.
In fact, by convexity of $f$,

$$\begin{aligned} y_2 > y_1 > x \implies & \frac{f(y_2) - f(x)}{y_2 - x} \geq \frac{f(y_1) - f(x)}{y_1 - x} \geq \partial_+ f(x) \\ \implies & f(y_2) - f(x) \geq \partial_+ f(x)(y_2 - x), \quad \forall y_2 > x \end{aligned}$$

Similarly, we will also have $f(y_2) - f(x) \geq \partial_+ f(x)(y_2 - x), \ \forall y_2 < x$.
Thus, this implies that $\partial_+ f(x) \in \partial f(x)$ and $\partial_+ f(x) + \varepsilon \notin \partial f(x)$, $\forall \varepsilon > 0$.
Secondly, we will see

$$\partial(f_1 + f_2)(x) = [\partial_-(f_1 + f_2)(x), \partial_+(f_1 + f_2)] = [\partial_- f_1(x) + \partial_- f_2(x), \partial_+ f_1(x) + \partial_+ f_2(x)]$$

□

# 2 Optimality condition of Convex Functions

## Recall: Euler's condition

1. If $f \in C^1(\mathbb{R}^n)$, and $x^*$ is a solution to $\min_{x \in \mathbb{R}^n} f(x)$, then $\nabla f(x^*) = 0$.
2. If $f(x)$ is convex, and $\nabla f(x^*) = 0$, then $x^*$ is an optimal solution to the problem $\min_{x \in \mathbb{R}^n} f(x)$.

**Proposition 2.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then*

$$\begin{aligned} & x \in \text{dom}(f) \textit{ be a local minimum of } f. \\ \iff & x \in \text{dom}(f) \textit{ is a global minimum of } f \\ \iff & 0 \in \partial f(x) \end{aligned}$$

 *Prepared by Max Shung*

*Proof.* 1. If $0 \in \partial f(x)$, then $f(y) \geq f(x) + \langle 0, y - x \rangle\,, \quad \forall y \in \mathbb{R}^n$.
This implies that $x$ is a global minimizer.

2. If $x$ is global minimum, then it is clear that $x$ is a local minimum.

3. If $x$ is a local minimum of $f$, then

$$f(x) \leq f(z), \quad \forall z \in B_\varepsilon(x)$$

Let $y \in \mathbb{R}^n$ be arbitrary, then for $\lambda > 0$ small enough such that $z = x + \lambda(y - x) \in B_\varepsilon(x)$. Then, we can deduce that

$$\begin{aligned} f(x) &\leq f\left(x + \lambda(y - x)\right) = f\left((1 - \lambda)x + \lambda y\right) \\ &\leq (1 - \lambda) f(x) + \lambda f(y) \qquad (\because f \text{ is convex}) \\ \implies \lambda f(y) &\geq \lambda f(x) + \underbrace{\langle 0, y - x \rangle}_{=0}, \quad \forall y \in \mathbb{R}^n \\ \implies f(y) &\geq f(x) \qquad (\because \lambda > 0) \end{aligned}$$

Thus, this implies that $0 \in \partial f(x)$.

□

*Remarks.* Let $X \subseteq \mathbb{R}^n$ be a convex set. We define

$$I_X(x) := \begin{cases} 0 & , \quad x \in X \\ +\infty & , \quad x \notin X \end{cases}$$

is a convex function, and the subdifferential of $I_X$ is

$$\partial I_X(x) = \{v \in \mathbb{R}^n : \langle v, y - x \rangle \leq 0, \quad \forall y \in X\}$$

for all $x \in X$. Alternatively, we say

$$\begin{aligned} v \in \partial I_X(x) &\iff I_X(y) \geq I_X(x) + \langle v, y - x \rangle\,, \quad \forall y \in \mathbb{R}^n \\ &\iff 0 \geq 0 + \langle v, y - x \rangle\,, \quad \forall y \in X \end{aligned}$$

**Proposition 3.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a **convex** function and $X = \operatorname{dom}(f)$. Then $x^* \in X$ be a solution to the problem*

$$\min_{x \in \mathbb{R}^n} f(x) = \min_{x \in X} f(x).$$

*if and only if there exists $v^* \in \partial f(x^*)$ such that $\langle v^*, y - x^* \rangle \geq 0, \quad \forall y \in X$.*

*Remarks.* In the past, we treat $X = \operatorname{dom}(f)$ is an optimization under constraint.

*Proof.* Note that

$$\begin{aligned} & x^* \text{ is a solution to } \min_{x \in X} f(x) \\ \iff & x^* \text{ is a solution to } \min_{x \in \mathbb{R}^n} f(x) + I_X(x) \\ \iff & 0 \in \partial\left(f + I_X\right)(x^*) \qquad \text{(by the previous proposition)} \end{aligned}$$

and $\partial(f + I_X)(x^*) = \partial f(x^*) + \partial I_X(x^*)$ by the property of subgradient.
This is equivalent to

$$\begin{aligned} &\iff \exists v \in \mathbb{R}^n \text{ such that } v^* \in \partial f(x^*) \text{ and } -v^* \in \partial I_X(x^*) \\ &\iff \exists v^* \in \partial f(x^*) \text{ and } \langle -v^*, y - x \rangle \leq 0, \quad \forall y \in X \\ &\iff \exists v^* \in \partial f(x^*) \text{ such that } \langle v^*, y - x \rangle \geq 0, \quad \forall y \in X \end{aligned}$$

and thus complete the proof. □

— End of Lecture 14 —